C428/39/112

Crack Analysis by Boundary Element Method

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Summary

A simple and efficient approach is proposed for the analysis of crack problems occuring in mechanical components widely used in the aerospace industry.

A linear stress analysis of the cracked components is performed using the engineering analysis system BEASY, based on the boundary element method, to derive accurate predictions of the stress intensity factors corresponding to either of the three modes K_I, K_{II} or K_{III}. With the proposed approach, the user has to model only the surface of the component, as opposed to its volume in finite elements. Thus, line elements in 2D and axisymmetric problems and surface quadrilaterals or triangles in 3D problems are used to mesh the components. The approach clearly results in substantial savings in mesh preparation effort. The paper will describe the application of the system to problems in the aerospace field and the convergence results which demonstrate the high accuracy and reliability of the technique.

Introduction

The prediction of cracks behaviour in aerospace components is a vital element in the design and maintenance effort of aircrafts. Experiment al investigate ions, analytical methods and numerical techniques have all been used and still are to assess the fracture resistance of a mechanical component or to predict the rate of propagation of an existing crack etc... The finite element method is the most general and perhaps the most commonly used approach for the solution of fracture mechanics problems. Albeit successful in predicting the overall behaviour of the structure, the finite element method is limited in that, unless a very high number of elements is used in the vicinity of the crack, it fails to accurately reproduce the stress field in that area. In addition, the method can be costly both in terms of the human effort required to build the finite element mesh and the computer ressources necessary to carry out the numerical solution.

The boundary element method (BEM)[1] has emerged in the late seventies as a strong alternative to the finite element method in a number of scientific fields. The BEM has introduced two useful concepts into the field of computational mechanics: that the problem under consideration can be fully and uniquely be defined by its boundary surface rather than its volume for the purpose of analysis and that integral equations can be used in conjunction with physically relevant weighting functions (fundamental solutions) to derive highly accurate results. In more practical terms, this means that the user of a boundary element program will only have to discretize the surface of the problem using line elements for 2D problems and quadrilateral or triangles for 3D problems. It also means that a relatively small number of elements will lead to a reasonable accuracy.

The advantages of the boundary element method in fracture mechanics problems have been recognized at an early stage of the method's development and practical applications of the method to fracture problems are now numerous[2]. In this paper however, an attempt is made to assess both the accuracy and convergence of a boundary element system (BEASY)[3] by analysing a number of practical fracture mechanics problems.

The Boundary Integral Equations of 3D Elastostatics

Boundary integral equations for 3D linear elastostatics can be written as follows[1]:

$$c_{ij}(P)u_i(P) + \int_s u_j(Q)t^*_{ij}(P,Q)dS = \int_s t_j(Q)u^*_{ij}(P,Q)dS +$$

$$\int_v u^*_{ij}(P,Q)b_j(Q)dV \qquad (1)$$

where S is the surface of the solid under consideration and V its volume. P and Q denote the source and field points, respectively. u_j and t_j refer to the displacement and traction fields (i and j are direction indices). u_i is the displacement at the source point P. u^*_{ij} and t^*_{ij} are the displacement and traction functions of the *Kelvin* point force solution. c_{ij} is a constant depending on the smoothness of the boundary at P and b_j are the body forces.

The volume integral containing the effect of body forces can be transformed into a surface one using a number of available techniques.

The integrals appearing in equations (1) can be numerically evaluated by dividing the boundary of the problem into a finite number of elements and by using standard integration procedures. Either displacements or tractions, in each of the three directions, will be given as boundary conditions to the problem. If equations (1) corresponding to each boundary node are formed, a linear system of equations of the following form is obtained:

$$[A](x)=(b)$$

where $[A]$ is a matrix of coefficients, (b) is a known vector and (x) is a vector containing the unknown tractions and displacements.

When the distance r between the source point P and the field point Q approaches zero, singularities of order $1/r$ and $1/r^2$ occur in equations (1). The standard gaussian integration scheme would then require a large number of gaussian points to achieve a reasonable accuracy or, ultimately, it may fail altogether. Special integration techniques, either analytical or numerical, are then used to resolve the singularities.

Once the main system of equations has been solved, solutions at selected points inside the volume of the model (internal points) can also be found by using integral equations similar to equations (1).

Boundary Element Analysis of Fracture Problems

Linear 2D, axisymmetric or 3D stress analyses can be performed to predict the stress intensity factors at a crack tip. The solid boundary element model of the fracture problem is built and the correct boundary conditions applied at either sides of the crack front. Stress intensity factors can then be calculated from the resulting displacement or stress field. A boundary element analysis of a fracture mechanics problems has, in addition to the essential reduction in the dimension of the problem, a number of other advantages that are perhaps no less important.

As shown in the previous section, both surface *tractions* and surface displacements can be taken as degrees *of freedom*, depending on the boundary conditions. This implies that boundary stresses are computed to a higher degree of accuracy than in Finite Elements where the differentiation of the displacement field induces unwanted inaccuracies in the stress calculation (stresses inside the domain of the problem can also be accurately computed in boundary elements if required). In addition, the use of exact fundamental solutions as weighting functions in the formulation of the problem improves the accuracy of the predictions and produces a faster convergence of the results.

Since the fracture mechanics solution is essentially a usual linear elastostatic analysis, *any crack shape* can be modelled using the standard pre-processing techniques. The *stress intensity factors for all 3* modes of fracture can be calculated by using, among other approaches, Irwin's equation based on either displacements or stresses[2]. Accurate estimates of the stress factor at the crack tip can in fact be obtained by extrapolating values from either the free or restrained surfaces of the crack depending on whether displacement-based or stress-based factors are used, respectively or using J integrals. In BEASY, the *stress intensity factors are automatically computed* by the post-processor at points selected by the user near the crack-

Finally, an important feature of boundary elements in general and BEASY in particular is the concept of *discontinuous elements*. When all, *or* part, of the element's nodes are inside the element rather than on its periphery, compatibility with adjacent elements is not satisfied and such an element is thus called discontinuous. These elements can very useful for two reasons. They can be used to model, with high accuracy, geometric or physical discontinuities such as the discontinuity of boundary conditions at the crack front. Secondly, such elements can be resorted to when nodes of adjacent elements do not coincide (e.g. linear element bordering quadratic one...). This gives more freedom to the user building the mesh since he does not have to ensure that the nodes coincide. The continuity pattern is devised automatically and, although it usually increases the number of degrees of freedom in the model. it has been found quite useful in improving the accuracy and saving engineer's time.

Applications

To illustrate the application of BEM to fracture mechanics problems, two crack problems representing circular tubes have been analysed by BEASY. In both cases, convergence studies have been performed to assess the solution quality.

Previous applications of BEM to fracture mechanics problems have demonstrated the solution to a number of different crack shapes and components. This paper focuses on edge cracks in two and three dimensional bodies.

2D analysis of a pressurized tube with an edge crack

The problem shown in figure 1 is a classical fracture mechanics problem where a circular tube, with an inner and outer radiuses of R_1 and R_2 respectively, is subjected to an internal uniform pressure p. A crack of length a has developed along one edge of the tube and is also loaded by uniform pressure p. Values of $R_1=1.$, $R_2=1.25$, $p=1.$, $a=0.125$ and a Poisson ratio of 0.3 have been used. The analytical stress intensity factor given by *Rooke and Cartwright*[5] is $K_I=6.43$. Using the problem's symmetry, a half-model of the tube cross-section has been built and four different boundary element models used to assess the convergence of the K_I stress intensity factor calculations at the crack tip. These results, presented in table 1, clearly demonstrate the convergence of the predictions and show that a relatively coarse boundary element mesh is required to achieve a reasonable accuracy.

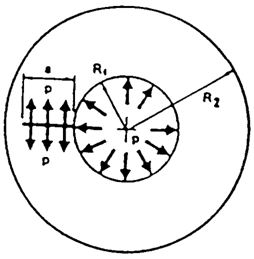


Figure 1a. Cracked tube under internal pressure: Loading

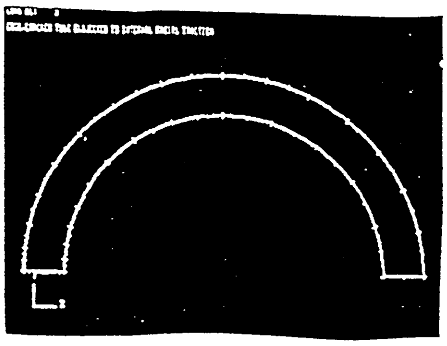


Figure 1b. Cracked tube under internal pressure: boundary element mesh.

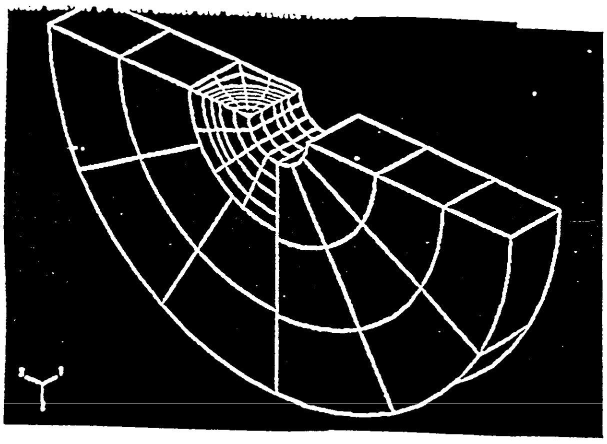
	Number of Elements	Number of Nodes	Beasy Prediction	% Error
Mesh 1	41	88	6.63	3.0%
Mesh 2	47	100	6.57	2.0%
Mesh 3	51	108	6.56	1.9%
Mesh 4	59	124	6.54	1.6%

Table 1. Predictions of K_I stress intensity factor of an internally pressurised tube with an edge crack.

3D model of a edge-cracked disc subject to external pressure

A quarter-circular flat crack of radius r has developed across the edge of a turbine disc of depth t, having an internal and external radiuses of R_1 and R_2, respectively. In the boundary element model shown in figure 2, one symmetry plane is used and a half-model built with dimensions $R_1=4$, $R_2=32$, $a=4$, $t=8$. The disc is subjected to an external uniform pressure $p=1$ applied at its outer periphery. Three different boundary element meshes, also shown in figure 2, have been used to derive predictions of the K_I stress intensity factors and assess their convergence by comparing to results presented by *Tan*[6]. Plane strain conditions were assumed around the crack front except at the edge where the crack front meets the free surface along the thickness where plane stress conditions were considered prevailing.

Table 2 compares results published by Tan and BEASY results for the 3 different meshes. Values of K_I are



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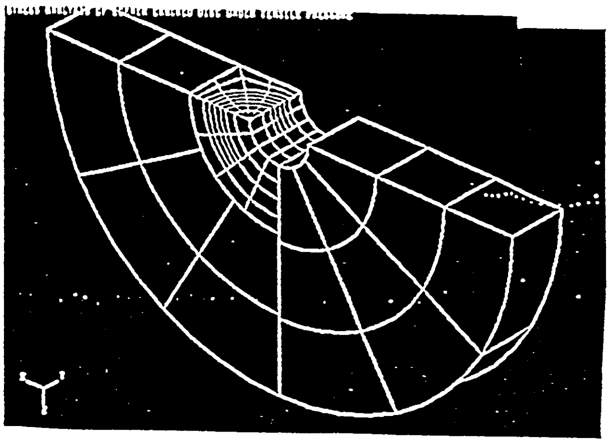

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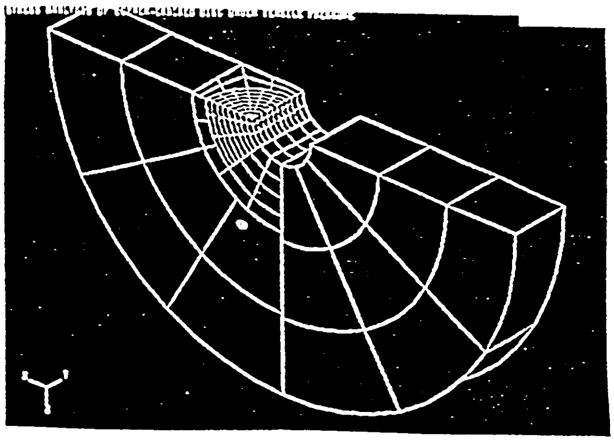

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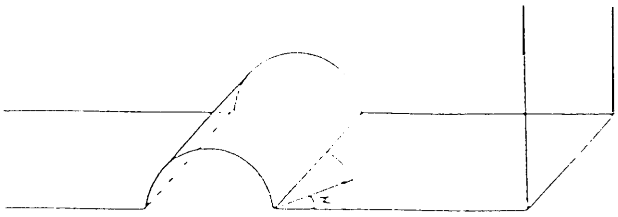

Figure 3. Angle α along crack front of cracked disc

shown at different positions around the crack front measured by angle z shown in figure 3. Meshes 1, 2 and 3 contain 113, 121, 226 elements and 299, 404, 742 nodes, respectively.

Angle z	Tan	BEASY Mesh 1	% Error	BEASY Mesh 2	% Error	BEASY Mesh 3	% Error
0	3.81	3.512	-7.9%	3.650	-4.3%	3.794	-0.5%
15	3.52	3.462	-1.7%	3.575	1.5%	3.505	-0.4%
30	3.41	3.333	-2.2%	3.378	-0.9%	3.351	-1.7%
45	3.34	3.268	-2.2%	3.281	-1.8%	3.297	-1.3%
60	3.39	3.297	-2.6%	3.354	-0.9%	3.347	-1.1%
90	3.79	3.773	-0.5%	3.724	-1.8%	3.790	0.0%

Table 2. Predictions of K_I stress intensity factor for an edge-cracked tube under external pressure.

Again, results shown in table 2 clearly demonstrate the convergence of BEASY results to those derived · Tan.

Conclusions

The linear elastostatic boundary element solutions of fracture mechanics problems lead to highly accurate and economical results. Practical problems have been solved using the boundary element analysis system BEASY and a steady convergence path has been observed when the numbers of nodes and elements were increased. The boundary element method can thus be considered to he an attractive alternative to the finite element method for the solution of fracture mechanics problems.

References

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